Application of the Faddeev Method to the Three-Spin Deviation Problem for the Heisenberg Model

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The three-spin deviation problem for the Heisenberg Hamiltonian is attacked by the Faddeev technique. The fact that the two-particle t matrix is separable facilitates the solution. In one dimension, the resulting equation can be numerically solved to obtain the eigenvalue. In two and three dimensions, integral equations in one vector variable appear, but no progress has so far been made towards their solution.

I. INTRODUCTION

EXACT eigenstates of the Heisenberg Hamiltonian

$$H=-\tfrac{1}{2}J\sum_{j,\delta}\mathbf{S}_j\cdot\mathbf{S}_{j+\delta} \tag{1}$$

have been investigated repeatedly. The summation in (1) extends over all lattice vectors $\mathbf{j}$ and over the vectors $\boldsymbol{\delta}$ which join one spin to one of its nearest neighbor. We shall consider the case where each spin has the magnitude $\frac{1}{2}$ and $J>0$. The total number of spins is N. Clearly, $S^z=\sum_j S_j^z$ is a good quantum number. It is convenient to classify the eigenstates in terms of the spin deviation n from the exactly aligned ground state with $S^z=-\frac{1}{2}N$. Define

$$n=S^z+\tfrac{1}{2}N. \tag{2}$$

$n=0$ corresponds to the ground state. The states with $n=1$ were called spin waves by Bloch.[1] Bethe[2] made a complete study of the various spin-deviation states of the Hamiltonian in a one-dimensional linear chain, but in two and three dimensions, the progress has been rather slow. As a part of his work on the general theory of spin-wave interactions, Dyson[3] investigated the two-spin deviation ($n=2$) problem in two and three dimensions. The solution was completed in these cases by Wortis[4] using the Green's functions, and by Fukuda and Wortis[5] by applying simply the Schrödinger equation. The two-spin deviation problem is soluble, because after separating the center-of-mass motion, it is reduced to an effective one-body problem. One can even solve this problem when the interaction extends beyond the nearest neighbors.[6]

The purpose of this work is to consider the problem for $n=3$ by the procedure developed by Faddeev[7] for three-particle scattering. The work of Wortis[4] and Fukuda and Wortis[5] shows that the two-body potential, albeit defined in an overcomplete basis, is separable. Therefore, the two-particle-scattering t matrix is also separable. It has been shown by Lovelace[8] that with a separable two-particle t matrix, the Faddeev equations can be simplified considerably.

We shall show that the Faddeev equations in one dimension can be reduced to a single integral equation, and we shall obtain the eigenvalue of the three-spin bound state, in which a block of three reversed spins travels together in a chain of aligned spins. The eigenvalue can be guessed—indeed, Bethe also guessed it—and verified numerically. The analytical solution of the integral equation, for instance, the eigenfunction, has not been obtained.

The Faddeev method is potentially capable of attacking the two- and three-dimensional problems, and in each case integral equations in one (vector) variable appear at the end. Unfortunately, however, no analytical method to handle these formidable integral equations has been found so far, and no progress has been made to extract the eigenvalues. Nevertheless, a soluble example of the Faddeev equations for the one-dimensional Hamiltonian is of sufficient interest in itself. Hopefully, the three-dimensional problem can be discussed in detail in future.

II. DYSON'S IDEAL SPIN-WAVE HAMILTONIAN

The two-spin derivation problem was solved by using an overcomplete basis,[9] and only in this way does a simplification occur. The problem of utilizing an overcomplete basis in connection with Eq. (1) was discussed extensively by Dyson. He found that the most convenient way was to introduce the ideal spin-wave model, and to define a pseudo-Hamiltonian in this model from which dynamical calculations might proceed. We briefly indicate the steps involved in the introduction of the ideal spin-wave Hamiltonian.

The commutation rules of the spin vector $\mathbf{S}_j$ attached to the lattice sites $\mathbf{j}$ are

$$[S_j^z,S_k^\pm]=\delta_{jk}S_j^\pm\,,\quad [S_j^+,S_k^-]=2\delta_{jk}S_j^z\,, \tag{3}$$

with

$$S_j^\pm=S_j^x\pm iS_j^y\,.$$

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The spin operators attached to the reciprocal lattice are defined by

$$\mathbf{S}_\lambda = N^{-1/2} \sum_j \exp(i\boldsymbol{\lambda}\cdot\mathbf{j})\mathbf{S}_j, \tag{4}$$

with commutation rules

$$\begin{aligned}[S_\lambda{}^z, S_\mu{}^\pm] &= \pm N^{-1/2} S_{\lambda+\mu}{}^\pm, \\ [S_\lambda{}^+, S_\mu{}^-] &= \pm N^{-1/2} S_{\lambda+\mu}{}^z.\end{aligned} \tag{5}$$

The fully aligned ground state of the system $|0\rangle$ is defined by

$$S_j{}^-|0\rangle = 0, \quad S_j{}^z|0\rangle = -\tfrac{1}{2}|0\rangle, \tag{6}$$

for all $\mathbf{j}$, or, equivalently,

$$S_\lambda{}^-|0\rangle = 0, \quad S_\lambda{}^z|0\rangle = -\tfrac{1}{2}N^{1/2}\delta_{\lambda 0}|0\rangle, \tag{7}$$

for all $\boldsymbol{\lambda}$. Bloch's spin wave with wave vector $\boldsymbol{\lambda}$ is defined by

$$|1_\lambda\rangle = S_\lambda{}^+|0\rangle. \tag{8}$$

These states are properly normalized and are orthogonal to each other and to $|0\rangle$. For states with more than one spin-wave present, we use a natural generalization of (8). Let $\{a\}$ represent any set of non-negative integers a_λ, one attached to each reciprocal-lattice vector $\boldsymbol{\lambda}$. Then the spin-wave state $|a\rangle$, containing a_λ spin waves with wave vectors $\boldsymbol{\lambda}$, is defined by

$$|a\rangle = \prod_\lambda [(a_\lambda!)^{-1/2}(S_\lambda{}^+)^{a_\lambda}]|0\rangle. \tag{9}$$

As soon as $\sum a_\lambda > 1$, the states (9) are neither normalized nor orthogonal to each other. The states (9) are much more numerous than the total number 2^N of independent states which the spin system possesses. The nonorthogonality of the states (9) produces an interaction between spin waves called the kinematical interaction. The physical cause of this interaction is the fact that for spin $\frac{1}{2}$, the spin at each site cannot be reversed more than once. There is another spin-wave interaction which arises from the fact that the Hamiltonian (1) is not diagonal in the states (9). This is called the dynamical interaction.

In terms of the operators (4), the Hamiltonian (1) can be written as

$$H = -\tfrac{1}{2}\sum_\lambda \gamma_\lambda \mathbf{S}_\lambda\cdot\mathbf{S}_{-\lambda}, \tag{10}$$

where

$$\gamma_\lambda = \sum_\delta \exp(i\boldsymbol{\delta}\cdot\boldsymbol{\lambda}). \tag{11}$$

We shall consider only the linear chain, the two-dimensional square, and the three-dimensional simple cubic lattice, each of which has inversion symmetry. The effect of operating with (10) on a state like (9) is given by the fomula

$$H|a\rangle = [E_0 + \sum_\lambda a_\lambda\epsilon_\lambda]|a\rangle + \sum_b Q_{ba}|b\rangle. \tag{12}$$

Here,

$$E_0 = -\tfrac{1}{8}JN\gamma_0, \quad \epsilon_\lambda = \tfrac{1}{2}J(\gamma_0 - \gamma_\lambda). \tag{13}$$

The second sum in (12) extends over spin-wave states $|b\rangle$ which are obtained from $|a\rangle$ by replacing one pair of spin waves $(\boldsymbol{\rho},\boldsymbol{\sigma})$ with a pair $(\boldsymbol{\rho}-\boldsymbol{\lambda}, \boldsymbol{\sigma}+\boldsymbol{\lambda})$. The Q_{ba} are numerical coefficients containing

$$\Gamma_{\rho\sigma}{}^\lambda = \gamma_\lambda - \gamma_{\lambda-\rho} - \gamma_{\lambda+\sigma} + \gamma_{\lambda+\sigma-\rho}. \tag{14}$$

The second term in (12) is the nondiagonal part representing a scattering of one spin wave by another.

Now we construct the ideal spin-wave model. To each lattice site $\mathbf{j}$, attach a harmonic oscillator which has states labeled by an integer u_j taking values from 0 to ∞. The oscillators possess creation operators $\eta_j{}^*$ and annihilation operators η_j satisfying the relations

$$\begin{aligned}[\eta_j,\eta_k] &= [\eta_j{}^*,\eta_k{}^*] = 0, \\ [\eta_j,\eta_k{}^*] &= \delta_{j\lambda}, \quad \eta_j{}^*\eta_j = u_j.\end{aligned} \tag{15}$$

A complete set of states for the whole system is

$$|u) = \prod_j [(u_j!)^{-1/2}(\eta_j{}^*)^{u_j}]|0). \tag{16}$$

The ideal spin-wave states are always indicated with round brackets. These are not only orthogonal but correctly normalized. Another complete set of orthogonal states is defined by

$$|a) = \prod_\lambda [(a_\lambda!)^{-1/2}(\alpha_\lambda{}^*)^{a_\lambda}]|0), \tag{17}$$

where

$$\alpha_\lambda{}^* = N^{-1/2}\sum_j \exp(i\boldsymbol{\lambda}\cdot\mathbf{j})\eta_j{}^*. \tag{18}$$

The $\alpha_\lambda{}^*$'s are creation operators for harmonic oscillators whose states are labeled by the integers a_λ; they satisfy the relations

$$[\alpha_\lambda,\alpha_\mu{}^*] = \delta_{\lambda\mu}, \quad \alpha_\lambda{}^*\alpha_\lambda = a_\lambda. \tag{19}$$

The harmonic oscillators, one attached to each point of the reciprocal lattice, we call ideal spin waves.

The physical spin-wave states $|a\rangle$ and the ideal spin-wave states $|a)$ are in one-to-one correspondence, but they belong to totally different Hilbert spaces. The states $|a)$ are orthogonal and kinematically independent, while the states $|a\rangle$ are not.

Dyson now constructs an operator H in the ideal Hilbert space which has the same effect on the ideal spin-wave state $|a)$ as the Hamiltonian in the physical Hilbert space has on the states $|a\rangle$. That is to say,

$$H|a) = [E_0 + \sum_\lambda a_\lambda\epsilon_\lambda]|a) + \sum_b Q_{ba}|b), \tag{20}$$

with the same numerical coefficients ϵ_λ and Q_{ba} which appear in (12). Such a Hamiltonian is given by

$$H = E_0 + \sum_\lambda \epsilon_\lambda\alpha_\lambda{}^*\alpha_\lambda - \tfrac{1}{4}JN^{-1}\sum_{\lambda\rho\sigma} \alpha_{\sigma+\lambda}{}^*\alpha_{\rho-\lambda}{}^*\alpha_\rho\alpha_\sigma\Gamma_{\rho\sigma}{}^\lambda. \tag{21}$$

This takes a particularly simple form when written in terms of the atomic oscillator coordinates (15):

$$H=E_0+\tfrac{1}{4}J\sum_{j\delta}(\eta_j^*-\eta_{j+\delta}^*)(\eta_j-\eta_{j+\delta})+\tfrac{1}{4}J\sum_{j\delta}\eta_j^*\eta_{j+\delta}^*(\eta_j-\eta_{j+\delta})^2. \quad (22)$$

The dynamical interaction in the ideal model still acts only between nearest neighbors. The last term in (22) is not Hermitian, and H cannot be directly interpreted as a Hamiltonian in the ideal model, as it could be in the physical model. Nevertheless, the scattering processes can be caclulated in the usual way, treating H as if it were an ordinary Hamiltonian and ignoring the distinction between physical and ideal models.

III. TWO-SPIN DEVIATION t MATRIX

Write the Hamiltonian as

$$H=H_1+H_2, \quad (23)$$

where H_1 is the first two terms of (21) or (22). H_2 is the dynamical interaction. H_1 is diagonal in the momentum space. The exact eigenvalues are measured with respect to E_0. We want to solve the Schrödinger equation

$$(H_1+H_2)\psi=E\psi \quad (24)$$

in the space $n=2$. Now

$$\psi=\sum_{jk}\psi(\mathbf{j},\mathbf{k})\eta_j^*\eta_k^*|0). \quad (25)$$

$\psi(\mathbf{j},\mathbf{k})=\psi(\mathbf{k},\mathbf{j})$ is the conventional two-particle wave function normalized, if necessary, such that

$$|\psi|^2=2\sum_{jk}|\psi(\mathbf{j},\mathbf{k})|^2=1. \quad (26)$$

Define an operator G such that

$$(H_1-E)G=1. \quad (27)$$

This is the usual Green's function, and we consider the matrix element of the operator G by inserting a complete set of states:

$$\tfrac{1}{2}\sum_{\mu_1,\mu_2}(\boldsymbol{\tau}_1\boldsymbol{\tau}_2|(H_1-E)|\boldsymbol{\mu}_1\boldsymbol{\mu}_2)(\boldsymbol{\mu}_1\boldsymbol{\mu}_2|G|\boldsymbol{\lambda}_1\boldsymbol{\lambda}_2)=(\boldsymbol{\tau}_1\boldsymbol{\tau}_2|\boldsymbol{\lambda}_1\boldsymbol{\lambda}_2). \quad (28)$$

The states $\boldsymbol{\mu}_1\neq\boldsymbol{\mu}_2$ are counted twice. The state with $\boldsymbol{\mu}_1=\boldsymbol{\mu}_2$ is counted once, but has a norm $=2$. Hence the factor $\frac{1}{2}$ takes proper counting of independent states into account. Hence we get

$$(\boldsymbol{\tau}_1\boldsymbol{\tau}_2|G|\boldsymbol{\lambda}_1\boldsymbol{\lambda}_2)=(\epsilon_{\tau_1}+\epsilon_{\tau_2}-E)^{-1}\times(\delta_{\tau_1\lambda_1}\delta_{\tau_2\lambda_2}+\delta_{\tau_1\lambda_2}\delta_{\tau_2\lambda_1}). \quad (29)$$

We now introduce the usual scattering equation with the t matrix:

$$\psi=\psi_a+Gt\psi_a. \quad (30)$$

Here, ψ_a represents the two free-spin waves, and

$$(H_1-E)\psi_a=0. \quad (31)$$

Hence,

$$t\psi_a=-H_2\psi. \quad (32)$$

Using (25), we obtain

$$H_2\psi=\tfrac{1}{4}J\sum_{j\delta}\eta_j^*\eta_{j+\delta}^*(\eta_j-\eta_{j+\delta})^2\psi$$
$$=J\sum_{j\delta}\eta_j^*\eta_{j+\delta}^*(0)[\psi(\mathbf{j},\mathbf{j})-\psi(\mathbf{j}+\boldsymbol{\delta},\mathbf{j})]. \quad (33)$$

As $\psi_a(\mathbf{i},\mathbf{j})$ we take two spin waves of wave vectors $\boldsymbol{\lambda}+\boldsymbol{\rho}$ and $\boldsymbol{\lambda}-\boldsymbol{\rho}_\mu$:

$$\psi_a(\mathbf{i},\mathbf{j})=N^{-1}\{\exp[i(\boldsymbol{\lambda}+\boldsymbol{\rho})\cdot\mathbf{i}]\exp[i(\boldsymbol{\lambda}-\boldsymbol{\rho})\cdot\mathbf{j}]+\exp[i(\boldsymbol{\lambda}+\boldsymbol{\rho})\cdot\mathbf{j}]\exp[i(\boldsymbol{\lambda}-\boldsymbol{\rho})\cdot\mathbf{i}]\}. \quad (34)$$

From (18) and (34), we get

$$t\psi_a=2t\alpha_{\lambda+\rho}^*\alpha_{\lambda-\rho}^*|0)\equiv 2t|\boldsymbol{\lambda}\boldsymbol{\rho}). \quad (35)$$

From (33) and (35), we obtain

$$(\boldsymbol{\lambda}\boldsymbol{\mu}|t\psi_a)=2(\boldsymbol{\lambda}\boldsymbol{\mu}|t|\boldsymbol{\lambda}\boldsymbol{\rho})$$
$$=-J\sum_{j\delta}(0|\alpha_{\lambda+\mu}\alpha_{\lambda-\mu}\eta_j^*\eta_{j+\delta}^*|0)\times[\psi(\mathbf{j},\mathbf{j})-\psi(\mathbf{j}+\boldsymbol{\delta},\mathbf{j})],$$

and with (18), we have

$$(\boldsymbol{\lambda}\boldsymbol{\mu}|t|\boldsymbol{\lambda}\boldsymbol{\rho})=-JN^{-1}\sum_{j\delta}\exp(-2i\boldsymbol{\lambda}\cdot\mathbf{j}-i\boldsymbol{\lambda}\cdot\boldsymbol{\delta})\times\cos(\boldsymbol{\mu}\cdot\boldsymbol{\delta})[\psi(\mathbf{j},\mathbf{j})-\psi(\mathbf{j}+\boldsymbol{\delta},\mathbf{j})]. \quad (36)$$

For ψ, we use (30), (34), and (35), calculate the matrix elements with (29), and we obtain

$$(\boldsymbol{\lambda}\boldsymbol{\mu}|t|\boldsymbol{\lambda}\boldsymbol{\rho})$$
$$=-2JN^{-1}\sum_{\delta}\cos(\boldsymbol{\mu}\cdot\boldsymbol{\delta})[\exp(-i\boldsymbol{\lambda}\cdot\boldsymbol{\delta})-\cos(\boldsymbol{\rho}\cdot\boldsymbol{\delta})]$$
$$-JN^{-1}\sum_{\delta\nu}\cos(\boldsymbol{\mu}\cdot\boldsymbol{\delta})[\exp(-i\boldsymbol{\lambda}\cdot\boldsymbol{\delta})-\exp(-i\boldsymbol{\nu}\cdot\boldsymbol{\delta})]\times[\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E]^{-1}(\boldsymbol{\lambda}\boldsymbol{\nu}|t|\boldsymbol{\lambda}\boldsymbol{\rho}). \quad (37)$$

We then take the different dimensions successively:

For one dimension (linear chain with spins unit distance apart),

$$(\lambda\mu|t|\lambda\rho)=-\frac{4J}{N}\cos\mu(\cos\lambda-\cos\rho)-\frac{2J}{N}\cos\mu\sum_{\nu}(\cos\lambda-\cos\nu)\times(\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E)^{-1}(\lambda\nu|t|\lambda\rho). \quad (38)$$

The solution is

$$(\lambda\mu|t|\lambda\rho)=\cos\mu\,\varphi(\lambda\rho), \quad (39)$$

with

$$\varphi(\lambda\rho)=-(4J/N)(\cos\lambda-\cos\rho)\Big/\left[1+\frac{2J}{\pi}\int_0^\pi\frac{\cos\nu(\cos\lambda-\cos\nu)}{\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E}d\nu\right]. \quad (40)$$

We have, from (13),

$$\epsilon_\lambda = J(1-\cos\lambda). \tag{41}$$

The integral in (40) can be done and the two-particle t matrix can be put in the form

$$(\lambda\mu|t|\lambda\rho) = -\frac{4J}{N}\frac{\cos\mu(\cos\lambda-\cos\rho)\cos^2\lambda[(1-z)^2-\cos^2\lambda]^{1/2}}{(1-z)\{[(1-z)^2-\cos^2\lambda]^{1/2}-(1-z-\cos^2\lambda)\}}, \tag{42}$$

where $z=E/2J$. The t matrix has branch points corresponding to the edges of the continuum $E=2J(1\pm\cos\lambda)$ and a bound-state pole at

$$E=\tfrac{1}{2}J(1-\cos 2\lambda). \tag{43}$$

Recall that the c.m. momentum of the two-spin bound complex is 2λ.

For two dimensions (square of unit side), we write

$$(\boldsymbol{\lambda}\boldsymbol{\mu}|t|\boldsymbol{\lambda}\boldsymbol{\rho}) = \cos\mu_x\varphi^1+\cos\mu_y\varphi^2, \tag{44}$$

and

$$\varphi^1=\frac{1}{\Delta}\begin{vmatrix} -\dfrac{4J}{N}(\cos\lambda_x-\cos\rho_x) & 2J\displaystyle\int\frac{(\cos\lambda_x-\cos\nu_x)\cos\nu_y}{\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E} \\ \dfrac{4J}{N}(\cos\lambda_y-\cos\rho_y) & 1+2J\displaystyle\int\frac{(\cos\lambda_y-\cos\nu_y)\cos\nu_y}{\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E}\end{vmatrix}, \tag{45}$$

$$\varphi^2=\frac{1}{\Delta}\begin{vmatrix} 1+2J\displaystyle\int\frac{(\cos\lambda_x-\cos\nu_x)\cos\nu_x}{\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E} & -\dfrac{4J}{N}(\cos\lambda_x-\cos\rho_x) \\ 2J\displaystyle\int\frac{(\cos\lambda_y-\cos\nu_y)\cos\nu_x}{\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E} & -\dfrac{4J}{N}(\cos\lambda_y-\cos\rho_y)\end{vmatrix}, \tag{46}$$

and the determinant Δ is given by

$$\Delta=\begin{vmatrix} 1+2J\displaystyle\int\frac{(\cos\lambda_x-\cos\nu_x)\cos\nu_x}{\epsilon_{\lambda-\nu}+\epsilon_{\lambda+\nu}-E} & 2J\displaystyle\int\frac{(\cos\lambda_x-\cos\nu_x)\cos\nu_y}{\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E} \\ 2J\displaystyle\int\frac{(\cos\lambda_y-\cos\nu_y)\cos\nu_x}{\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E} & 1+2J\displaystyle\int\frac{(\cos\lambda_y-\cos\nu_y)\cos\nu_y}{\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E}\end{vmatrix}. \tag{47}$$

Here,

$$\int\equiv\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\nu_x d\nu_y,$$

and

$$\epsilon_\lambda=J(2-\cos\lambda_x-\cos\lambda_y). \tag{48}$$

A detailed discussion of the integrals is given by Wortis.[4]

For three dimensions (simple cubic lattice), the form of the two-particle t matrix is

$$(\boldsymbol{\lambda}\boldsymbol{\mu}|t|\boldsymbol{\lambda}\boldsymbol{\rho}) = [\cos\mu_x\varphi^1+\cos\mu_y\varphi^2+\cos\mu_z\varphi^3]/\Delta. \tag{49}$$

Define a determinant

$$\Delta=\begin{vmatrix} 1+2J\displaystyle\int\frac{(\cos\lambda_x-\cos\nu_x)\cos\nu_x}{B} & 2J\displaystyle\int\frac{(\cos\lambda_x-\cos\nu_x)\cos\nu_y}{B} & 2J\displaystyle\int\frac{(\cos\lambda_x-\cos\nu_x)\cos\nu_z}{B} \\ 2J\displaystyle\int\frac{(\cos\lambda_y-\cos\nu_y)\cos\nu_x}{B} & 1+2J\displaystyle\int\frac{(\cos\lambda_y-\cos\nu_y)\cos\nu_y}{B} & 2J\displaystyle\int\frac{(\cos\lambda_y-\cos\nu_y)\cos\nu_z}{B} \\ 2J\displaystyle\int\frac{(\cos\lambda_z-\cos\nu_z)\cos\nu_x}{B} & 2J\displaystyle\int\frac{(\cos\lambda_z-\cos\nu_z)\cos\nu_y}{B} & 1+2J\displaystyle\int\frac{(\cos\lambda_z-\cos\nu_z)\cos\nu_z}{B}\end{vmatrix}. \tag{50}$$

Here,

$$\int \equiv \frac{1}{8\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\nu_x d\nu_y d\nu_z\,,$$

$$B=\epsilon_{\lambda+\nu}+\epsilon_{\lambda-\nu}-E\,,$$

and

$$\epsilon_\lambda = J(3-\cos\lambda_x-\cos\lambda_y-\cos\lambda_z)\,. \tag{51}$$

φ^1, φ^2, and φ^3 can be written as determinants by replacing the first, second, and third columns, respectively, with the column

$$[-(4J/N)(\cos\lambda_x-\cos\rho_x)\,,\quad -(4J/N)(\cos\lambda_y-\cos\rho_y)\,,\quad -(4J/N)(\cos\lambda_z-\cos\rho_z)].$$

Wortis[4] gives a discussion of the bound states, the poles outside the continuum, for this problem. However, the integrals in (50) are formidable, and have not been analytically obtained. Certain obvious symmetries exist among the φ's; they are useful in the reduction of the integral equations in the three-spin deviation problem.

IV. THREE-SPIN DEVIATION PROBLEM

We have demonstrated the two-particle t matrix is separable in one, two, and three dimensions. However, in the two latter cases, the Faddeev equations lead to integral equations for which no successful method of solution has been found so far. We shall therefore concentrate on the one-dimensional problem. The reader is invited to check hereafter that, except in the final solution of Sec. V, the dimensionality is of no consequence.

We want to solve Eq. (24) for $n=3$, and E now stands for three-particle energy. Put

$$\psi=\sum_{ijk}\psi(\mathbf{ijk})\eta_i^*\eta_j^*\eta_k^*|0)\,, \tag{52}$$

where $\psi(\mathbf{ijk})$ is the completely symmetric three-particle wave function in the conventional sense. Define the Green's function G_0 as before

$$(H_1-E)G_0=1\,. \tag{53}$$

Putting

$$\psi=\psi_a+G_0T\psi_a\,, \tag{54}$$

with

$$\psi_a(\mathbf{ijk})=N^{-3/2}\sum_P \exp(i\lambda_{P_1}\cdot\mathbf{i}+i\lambda_{P_2}\cdot\mathbf{j}+i\lambda_{P_3}\cdot\mathbf{k})\,, \tag{55}$$

with P being any permutation of numbers 1, 2, and 3 (symmetric group S_3), we get

$$T\psi_a=-H_2\psi \tag{56}$$

$$\begin{aligned}\equiv -\tfrac{1}{2}J\sum_{l\delta k}\eta_l^*\eta_{l+\delta}^*\eta_k^*|0)[\psi(\mathbf{ljk})+\psi(\mathbf{l+\delta l+\delta k})-\psi(\mathbf{ll+\delta k})-\psi(\mathbf{l+\delta lk})]\\ -\tfrac{1}{2}J\sum_{l\delta j}\eta_l^*\eta_{l+\delta}^*\eta_j^*|0)[\psi(\mathbf{ljl})+\psi(\mathbf{l+\delta jl+\delta})-\psi(\mathbf{ljl+\delta})-\psi(\mathbf{l+\delta j}l)]\\ -\tfrac{1}{2}J\sum_{li\delta}\eta_l^*\eta_{l+\delta}^*\eta_i^*|0)[\psi(\mathbf{ill})+\psi(\mathbf{il+\delta l+\delta})-\psi(\mathbf{ill+\delta})-\psi(\mathbf{il+\delta l})].\end{aligned} \tag{57}$$

Let us denote the three-momentum vectors $(\boldsymbol{\tau}_1\boldsymbol{\tau}_2\boldsymbol{\tau}_3)$ by $[\tau]$. Then we want to compute

$$([\mu]|T\psi_i)\equiv 3!([\mu]|T|[\lambda])=(0|\alpha_{\mu_3}\alpha_{\mu_2}\alpha_{\mu_1}H_2\psi)\,. \tag{58}$$

The reduction of the right-hand side requires rather tedious but completely straightforward algebra. In one dimension, we get

$$\begin{aligned}([\mu]|T|[\lambda])=-4JN^{-1}\Big\{&\sum_{1,2,3}\delta(\mu_1-\lambda_1)\delta(\mu_2+\mu_3-\lambda_2-\lambda_3)\cos\tfrac{1}{2}(\mu_2-\mu_3)[\cos\tfrac{1}{2}(\mu_2+\mu_3)-\cos\tfrac{1}{2}(\lambda_2-\lambda_3)]\\ &+\sum_{1,2,3}\delta(\mu_2-\lambda_1)\delta(\mu_3+\mu_1-\lambda_2-\lambda_3)\cos\tfrac{1}{2}(\mu_3-\mu_1)[\cos\tfrac{1}{2}(\mu_3+\mu_1)-\cos\tfrac{1}{2}(\lambda_2-\lambda_3)]\\ &+\sum_{1,2,3}\delta(\mu_3-\lambda_1)\delta(\mu_1+\mu_2-\lambda_2-\lambda_3)\cos\tfrac{1}{2}(\mu_1-\mu_2)[\cos\tfrac{1}{2}(\mu_1+\mu_2)-\cos\tfrac{1}{2}(\lambda_2-\lambda_3)]\Big\}\\ &-\tfrac{2}{3}JN^{-1}\sum_{[\tau]}\Big\{\sum_{1,2,3}\delta(\mu_1-\tau_1)\delta(\mu_2+\mu_3-\tau_2-\tau_3)\cos\tfrac{1}{2}(\mu_2-\mu_3)[\cos\tfrac{1}{2}(\mu_2+\mu_3)-\cos\tfrac{1}{2}(\tau_2-\tau_3)]\\ &+\sum_{1,2,3}\delta(\mu_2-\tau_1)\delta(\mu_3+\mu_1-\tau_2-\tau_3)\cos\tfrac{1}{2}(\mu_3-\mu_1)[\cos\tfrac{1}{2}(\mu_3+\mu_1)-\cos\tfrac{1}{2}(\tau_2-\tau_3)]\\ &+\sum_{1,2,3}\delta(\mu_3-\tau_1)\delta(\mu_1+\mu_2-\tau_2-\tau_3)\cos\tfrac{1}{2}(\mu_1-\mu_2)[\cos\tfrac{1}{2}(\mu_1+\mu_2)-\cos\tfrac{1}{2}(\tau_2-\tau_3)]\Big\}\\ &\times(\epsilon_{\tau_1}+\epsilon_{\tau_2}+\epsilon_{\tau_3}-E)^{-1}([\tau]|T|[\lambda])\,.\end{aligned} \tag{59}$$

The symbol $\sum_{1,2,3}$ implies a sum over terms with cyclic permutation over $(\lambda_1\lambda_2\lambda_3)$ and $(\tau_1\tau_2\tau_3)$. For example, the first term of the first part is

$$\delta(\mu_1-\lambda_1)\delta(\mu_2+\mu_3-\lambda_2-\lambda_3)\cos\tfrac{1}{2}(\mu_2-\mu_3)[\cos\tfrac{1}{2}(\mu_2+\mu_3)-\cos\tfrac{1}{2}(\lambda_2-\lambda_3)]$$
$$+\delta(\mu_1-\lambda_2)\delta(\mu_2+\mu_3-\lambda_3-\lambda_1)\cos\tfrac{1}{2}(\mu_2-\mu_3)[\cos\tfrac{1}{2}(\mu_2+\mu_3)-\cos\tfrac{1}{2}(\lambda_3-\lambda_1)]$$
$$+\delta(\mu_1-\lambda_3)\delta(\mu_2+\mu_3-\lambda_1-\lambda_2)\cos\tfrac{1}{2}(\mu_2-\mu_3)[\cos\tfrac{1}{2}(\mu_2+\mu_3)-\cos\tfrac{1}{2}(\lambda_1-\lambda_2)]. \quad (60)$$

Taking advantage of the symmetry of the energy denominator and symmetric nature of $|[\tau])$, the second part can be simplified into

$$-2JN^{-1}\sum_{[\tau]}\{\delta(\mu_1-\tau_1)\delta(\mu_2+\mu_3-\tau_2-\tau_3)\cos\tfrac{1}{2}(\mu_2-\mu_3)[\cos\tfrac{1}{2}(\mu_2+\mu_3)-\cos\tfrac{1}{2}(\tau_2-\tau_3)]$$
$$+\delta(\mu_2-\tau_2)\delta(\mu_3+\mu_1-\tau_3-\tau_1)\cos\tfrac{1}{2}(\mu_3-\mu_1)[\cos\tfrac{1}{2}(\mu_3+\mu_1)-\cos\tfrac{1}{2}(\tau_3-\tau_1)]$$
$$+\delta(\mu_3-\tau_3)\delta(\mu_1+\mu_2-\tau_1-\tau_2)\cos\tfrac{1}{2}(\mu_1-\mu_2)[\cos\tfrac{1}{2}(\mu_1+\mu_2)-\cos\tfrac{1}{2}(\tau_1-\tau_2)]\}$$
$$\times(\epsilon_{\tau_1}+\epsilon_{\tau_2}+\epsilon_{\tau_3}-E)^{-1}\cdot([\tau]|T|[\lambda]). \quad (61)$$

This form is useful later. Clearly in (59), an over-all momentum-conserving δ function can be taken out

$$([\mu]|T|[\lambda])=\delta(\mu_1+\mu_2+\mu_3-\lambda_1-\lambda_2-\lambda_3)([\mu]'|T|[\lambda]'). \quad (62)$$

The prime denotes that the set $[\mu]'$ is such that $\mu_1+\mu_2+\mu_3$ is a constant.

For further reduction it is convenient to introduce the following variables, the analog of Lovelace's variables[8]:

$$\begin{aligned} &K=\mu_1+\mu_2+\mu_3,\\ &p_1=\mu_2+\mu_3,\quad p_2=\mu_3+\mu_1,\quad p_3=\mu_1+\mu_2,\\ &q_1=\tfrac{1}{2}(\mu_2-\mu_3),\quad q_2=\tfrac{1}{2}(\mu_3-\mu_1),\quad q_3=\tfrac{1}{2}(\mu_1-\mu_2).\end{aligned} \quad (63)$$

Similarly,

$$\begin{aligned} &K=\lambda_1+\lambda_2+\lambda_3=\tau_1+\tau_2+\tau_3,\\ &\Lambda^3=\lambda_1+\lambda_2,\quad \theta^3=\tau_1+\tau_2,\\ &\Lambda_3=\tfrac{1}{2}(\lambda_1-\lambda_2),\quad \theta_3=\tfrac{1}{2}(\tau_1-\tau_2),\end{aligned} \quad (64)$$

and other relations obtained cyclically. Thus

$$([\mu]'|T|[\lambda]')=-4JN^{-1}[\sum_{1,2,3}\cos q_1\delta(p_1-\Lambda^1)(\cos\tfrac{1}{2}p_1-\cos\Lambda_1)+\sum_{1,2,3}\cos q_2\delta(p_2-\Lambda^1)(\cos\tfrac{1}{2}p_2-\cos\Lambda_1)$$
$$+\sum_{1,2,3}\cos q_3\delta(p_3-\Lambda^1)(\cos\tfrac{1}{2}p_3-\cos\Lambda_1)]-\tfrac{2}{3}JN^{-1}\sum_{[\theta]'}[\sum_{1,2,3}\cos q_1\delta(p_1-\theta^1)(\cos\tfrac{1}{2}p_1-\cos\theta_1)+\sum_{1,2,3}\cos q_2\delta(p_2-\theta^1)$$
$$\times(\cos\tfrac{1}{2}p_2-\cos\theta_1)+\sum_{1,2,3}\cos q_3\delta(p_3-\theta^1)(\cos\tfrac{1}{2}p_3-\cos\theta_1)][\epsilon(k\theta)-E]^{-1}([\theta]'|T|[\lambda]'). \quad (65)$$

The energy denominator can be expressed in any of the three sets of variables (θ^i,θ_i), whichever is convenient:

$$\begin{aligned}\epsilon(k\theta)-E&=3J-J\cos(k-\theta^1)-2J\cos\tfrac{1}{2}\theta^1\cos\theta_1-E\\ &=3J-J\cos(k-\theta^2)-2J\cos\tfrac{1}{2}\theta^2\cos\theta_2-E\\ &=3J-J\cos(k-\theta^3)-2J\cos\tfrac{1}{2}\theta^3\cos\theta_3-E.\end{aligned} \quad (66)$$

Now, following Faddeev and Lovelace, we write T as a sum of three different T matrices:

$$([\mu]'|T|[\lambda]')={}_1(Kp_1q_1|T^1|[\lambda]')+{}_2(Kp_2q_2|T^2|[\lambda]')+{}_3(Kp_3q_3|T^3|[\lambda]'). \quad (67)$$

In T^1 we use the first set of Lovelace variables, in T^2 the second set, and in T^3 the third set as indicated by the subscript. We can now write down the matrix element of the two-body interaction V_1 acting between particles 2 and 3:

$${}_1(Kp_1q_1|V_1|[\lambda]')=-(4J/N)\cos q_1[\delta(p_1-\Lambda^1)(\cos\tfrac{1}{2}p_1-\cos\Lambda_1)+\delta(p_1-\Lambda^2)$$
$$\times(\cos\tfrac{1}{2}p_1-\cos\Lambda_2)+\delta(p_1-\Lambda^3)(\cos\tfrac{1}{2}p_1-\cos\Lambda_3)]. \quad (68)$$

Similarly, V_2 and V_3 are defined in an obvious fashion. Equation (65) can now be broken up into three equations for T^1, T^2, T^3:

$$T^1=V_1+V_1G_0T,\quad T^2=V_2+V_2G_0T,\quad T^3=V_3+V_3G_0T, \quad (69)$$

where the operator G_0 is given by

$$G_0=\tfrac{1}{6}\sum_{[\tau]}|\tau_1\tau_2\tau_3)[1/\epsilon(\tau_1\tau_2\tau_3)-E](\tau_1\tau_2\tau_3|. \quad (70)$$

The sum of the τ's equals K. We can check that this is exactly the operator (53), the factor $\frac{1}{6}$ taking proper counting and normalization into account. The expanded equation for T^1 is

$$_1(Kp_1q_1|T^1|[\lambda]')=-(4J/N)\cos q_1[\delta(p_1-\Lambda^1)(\cos\tfrac{1}{2}p_1-\cos\Lambda_1)+\delta(p_1-\Lambda^2)(\cos\tfrac{1}{2}p_1-\cos\Lambda_2)+\delta(p_1-\Lambda^3)$$
$$\times(\cos\tfrac{1}{2}p_1-\cos\Lambda_3)]-(2J/N)\cos q_1\sum_{\theta^1,\theta_1}\delta(p_1-\theta^1)(\cos\tfrac{1}{2}p_1-\cos\theta_1)[\epsilon(k\theta^1\theta_1)-E]^{-1}{}_1(k\theta^1\theta_1|T|[\lambda]'). \quad (71)$$

Now Faddeev's method eliminates the interactions V_1, V_2, V_3 in favor of the corresponding t matrices defined as follows:

$$T_1=V_1+V_1G_0T_1,\quad T_2=V_2+V_2G_0T_2,\quad T_3=V_3+V_3G_0T_3. \quad (72)$$

Now from (69) and (72),

$$V_1=T_1(1+G_0T_1)^{-1},\quad V_1=T^1(1+G_0T)^{-1}.$$

Taking the inverses,

$$V_1^{-1}=(1+G_0T_1)(T_1)^{-1}=(T_1)^{-1}+G_0\equiv(1+G_0T)(T^1)^{-1}.$$

Hence

$$[(T_1)^{-1}+G_0]T^1=1+G_0T,$$

and multiplying by T_1 on the left,

$$T^1+T_1G_0T^1=T_1+T_1G_0T,$$

so that

$$T^1=T_1+T_1G_0(T^2+T^3). \quad (73a)$$

Similarly,

$$T^2=T_2+T_2G_0(T^3+T^1), \quad (73b)$$

$$T^3=T_3+T_3G_0(T^1+T^2). \quad (73c)$$

Equations (73) represent the Faddeev equations. Now by (72),

$$_1(Kp_1q_1|T_1|[\lambda]')={}_1(Kp_1q_1|V_1|[\lambda]')-\frac{2J}{N}\sum_{\theta^1,\theta_1}\frac{\cos q_1\delta(p_1-\theta^1)(\cos\frac{1}{2}p_1-\cos\theta_1)}{3J-J\cos(k-\theta^1)-2J\cos\frac{1}{2}\theta^1\cos\theta_1-E}{}_1(k\theta^1\theta_1|T_1|[\lambda]')$$
$$\equiv{}_1(Kp_1q_1|V_1|[\lambda]')-\frac{2J}{N}\sum_{\theta_1}\frac{\cos q_1(\cos\frac{1}{2}p_1-\cos\theta_1){}_1(Kp_1q_1|T_1|[\lambda]')}{3J-J\cos(K-p_1)-2J\cos\frac{1}{2}p_1\cos\theta_1-E}. \quad (74)$$

The solution is clearly

$$_1(Kp_1q_1|T_1|[\lambda]')={}_1(Kp_1q_1|V_1|[\lambda]')\bigg/\left(1+\frac{2J}{N}\sum_{\theta}\frac{(\cos\frac{1}{2}p_1-\cos\theta)\cos\theta}{3J-J\cos(K-p_1)-2J\cos\frac{1}{2}p_1\cos\theta-E}\right). \quad (75)$$

Compare this with Eqs. (39) and (40). The integral can be done and we have

$$_1(Kp_1q_1|T_1|[\lambda]')={}_1(Kp_1q_1|V_1|[\lambda]')\cos^2\tfrac{1}{2}p_1/D, \quad (76)$$

where

$$D=1-\omega+\tfrac{1}{2}[1-\cos(K-p_1)]-\{1-\omega+\tfrac{1}{2}[1-\cos(K-p_1)]-\cos^2\tfrac{1}{2}p_1\}$$
$$\times\left(1-\frac{\cos^2\frac{1}{2}p_1}{\{1-\omega+\frac{1}{2}[1-\cos(K-p_1)]\}^2}\right)^{-1/2}, \quad (77)$$

where $\omega=E/2J$. Using (68), we write

$$_1(Kp_1q_1|T_1|y)=\cos q_1\Phi_1(Kp_1y),\quad {}_2(Kp_2q_2|T_2|y)=\cos q_2\Phi_2(Kp_2y),$$
$$_3(Kp_3q_3|T_3|y)=\cos q_3\Phi_3(Kp_3y). \quad (78)$$

Here Φ_1 is identified from (76) and (77). Notice that Φ_2 is the same function of p_2 as Φ_1 is of p_1 or Φ_3 is of p_3. The solutions of (73) are

$$_1(Kp_1q_1|T^1|y)={}_1(Kp_1q_1|T_1|y)+\cos q_1\Psi_1(Kp_1y),$$
$$_2(Kp_2q_2|T^2|y)={}_2(Kp_2q_2|T_2|y)+\cos q_2\Psi_2(Kp_2y), \quad (79)$$
$$_3(Kp_3q_3|T^3|y)={}_3(Kp_3q_3|T_3|y)+\cos q_3\Psi_3(Kp_3y).$$

As with Φ, Ψ_1 is the same function of p_1 as Ψ_2 or p_2 or Ψ_3 of p_3. Eq. (73) gives us for T^1

$$\cos q_1\Psi_1(Kp_1y)=\tfrac{1}{2}\sum_{p_2,q_2}\frac{{}_1(Kp_1q_1|T_1|Kp_2q_2)_2}{\epsilon(Kp_2q_2)-E}\cdot[{}_2(Kp_2q_2|T_2|y)+\cos q_2\Psi_2(Kp_2y)]$$
$$+\tfrac{1}{2}\sum_{p_3,q_3}\frac{{}_1(Kp_1q_1|T_1|Kp_3q_3)_3}{\epsilon(Kp_3q_3)-E}[{}_3(Kp_3q_3|T_3|y)+\cos q_3\Psi_3(Kp_3y)]. \quad (80$$

From (68) and (76), there are three terms in T_1, each of which gives the same contribution by symmetry. We choose to express each of them in terms of the set Kp_2q_2 for T^2 and Kp_3q_3 for T^3, so that a factor of 3 appears, and the final factor is $\frac{1}{2}$ in place of $\frac{1}{6}$. Now to calculate the matrix element of T_1, we have to reexpress Kp_2q_2 and Kp_3q_3 in terms of the first set of variables in (63). The formulas for kinematical transformation are

$$p_1=K-\tfrac{1}{2}p_2+q_2=K-\tfrac{1}{2}p_3-q_3,\quad q_1=\tfrac{1}{2}(K-\tfrac{3}{2}p_2-q_2)=\tfrac{1}{2}(\tfrac{3}{2}p_3-K-q_3). \tag{81}$$

Also we write

$$|Kp_2q_2)_2\equiv|K\bar{p}_2\bar{q}_2)_1,\quad |Kp_3q_3)_3\equiv|K\bar{p}_3\bar{q}_3)_1. \tag{82}$$

Eq. (80) becomes

$$\cos q_1\Psi_1(Kp_1y)=\tfrac{1}{2}\sum_{p_3,q_3}\frac{{}_1(Kp_1q_1|T_1|K\bar{p}_2\bar{q}_2)_1}{\epsilon(Kp_2q_2)-E}[{}_2(Kp_2q_2|T_2|y)+\cos q_2\Psi_2(Kp_2y)]$$
$$+\tfrac{1}{2}\sum_{p_2,q_2}\frac{{}_1(Kp_1q_1|T_1|K\bar{p}_3\bar{q}_3)_1}{\epsilon(Kp_3q_3)-E}[{}_3(Kp_3q_3|T_3|y)+\cos q_3\Psi_3(Kp_3y)]. \tag{83}$$

Now, recalling what is said under Eq. (80), we have

$${}_1(Kp_1q_1|T_1|K\bar{p}_2\bar{q}_2)_1=-(4J/N)\cos q_1\delta(p_1-\bar{p}_2)(\cos\tfrac{1}{2}p_1-\cos\bar{q}_2)\cos^2\tfrac{1}{2}p_1/D, \tag{84}$$
$${}_1(Kp_1q_1|T_1|K\bar{p}_3\bar{q}_3)_1=-(4J/N)\cos q_1\delta(p_1-\bar{p}_3)(\cos\tfrac{1}{2}p_1-\cos\bar{q}_3)\cos^2\tfrac{1}{2}p_1/D. \tag{85}$$

Canceling the common factor $\cos q_1$, Eq. (83) becomes

$$\Psi_1(Kp_1y)=\chi(p_1y)-\frac{2J}{N}\sum_{p_2,q_3}\frac{\delta(p_1-\bar{p}_2)(\cos\frac{1}{2}p_1-\cos\bar{q}_2)\cos^2\frac{1}{2}p_1\cos q_2}{D[\epsilon(Kp_2q_2)-E]}$$
$$-\frac{2J}{N}\sum_{p_3,q_3}\frac{\delta(p_1-\bar{p}_3)(\cos\frac{1}{2}p_1-\cos\bar{q}_3)\cos^2\frac{1}{2}p_1\cos q_3}{D[\epsilon(Kp_3q_3)-E]}\Psi_3(Kp_3y). \tag{86}$$

The inhomogeneous term is

$$\chi(p_1y)=-\frac{2J}{N}\sum_{p_2,q_2}\frac{\delta(p_1-\bar{p}_2)(\cos\frac{1}{2}p_1-\cos\bar{q}_2)\cos^2\frac{1}{2}p_1}{D[\epsilon(Kp_2q_2)-E]}{}_2(Kp_2q_2|T_2|y)$$
$$-\frac{2J}{N}\sum_{p_3,q_3}\frac{\delta(p_1-\bar{p}_3)(\cos\frac{1}{2}p_1-\cos\bar{q}_3)\cos^2\frac{1}{2}p_1}{D[\epsilon(Kp_3q_3)-E]}{}_3(Kp_3q_3|T_3|y). \tag{87}$$

For any given initial state $|y)$, the inhomogeneous term can be calculated. The δ functions can be removed by using (81):

$$\Psi_1(Kp_1y)=\chi(p_1y)-\frac{2J}{N}\sum_{p_2,q_2}\frac{\delta(p_1-K+\frac{1}{2}p_2-q_2)[\cos\frac{1}{2}p_1-\cos(\frac{1}{2}K-\frac{3}{4}p_2-\frac{1}{2}q_2)]\cos^2\frac{1}{2}p_1}{D[3J-J\cos(K-p_2)-2J\cos\frac{1}{2}p_2\cos q_2-E]}\cos q_2\Psi_2(Kp_2y)$$
$$-\frac{2J}{N}\sum_{p_3,q_3}\frac{\delta(p_1-K-\frac{1}{2}p_3+q_3)[\cos\frac{1}{2}p_1-\cos(\frac{3}{4}p_3-\frac{1}{2}K-\frac{1}{2}q_3)]\cos^2\frac{1}{2}p_1}{D[3J-J\cos(K-p_3)-2J\cos\frac{1}{2}p_3\cos q_3-E]}\cos q_3\Psi_3(Kp_3y), \tag{88}$$

or, dropping henceforth the unnecessary indices K,y:

$$\Psi_1(p_1)=\chi(p_1)-\frac{2J}{N}\sum_{p_2}\frac{[\cos\frac{1}{2}p_1-\cos(K-\frac{1}{2}p_1-p_2)]\cos(K-p_1-\frac{1}{2}p_2)\cos^2\frac{1}{2}p_1\Psi_2(p_2)}{D[3J-J\cos(K-p_2)-2J\cos\frac{1}{2}p_2\cos(K-p_1-\frac{1}{2}p_2)-E]}$$
$$-\frac{2J}{N}\sum_{p_3}\frac{[\cos\frac{1}{2}p_1-\cos(K-\frac{1}{2}p_1-p_3)]\cos(K-p_1-\frac{1}{2}p_3)\cos^2\frac{1}{2}p_1\Psi_3(p_3)}{D[3J-J\cos(K-p_3)-2J\cos\frac{1}{2}p_3\cos(K-p_1-\frac{1}{2}p_3)-E]}. \tag{89}$$

Because of the nature of Ψ_i, Eq. (89) represents a single integral equation for an unknown function Ψ, the least two terms becoming identical:

$$\Psi(p_1)=\chi(p_1)+\frac{4J}{N}\sum_{p_2}\frac{[\cos\frac{1}{2}p_1-\cos(K-\frac{1}{2}p_1-p_2)]\cos(K-p_1-\frac{1}{2}p_2)\cos^2\frac{1}{2}p_1\Psi(p_2)}{D[E-3J+J\cos(K-p_2)+2J\cos\frac{1}{2}p_2\cos(K-p_1-\frac{1}{2}p_2)]}. \tag{90}$$

This equation is obtained by starting from T^1; the same equation is obtained from T^2 and T^3 of (73b) and (73c). This follows from the nature of the Ψ's. More basically, this can be traced to the complete symmetry between the three-spin waves or the Bose statistics of the spin waves. The situation is analogous to that treated by Ahmadzadeh and Tjon for three pions constituting a bound particle.[10]

The three-spin-wave bound states are obtained from the homogeneous equation

$$\Psi(p_1)=\frac{1}{\pi}\int_{-\pi}^{\pi}dp_2\frac{[\cos\frac{1}{2}p_1-\cos(K-\frac{1}{2}p_1-p_2)]\cos(K-p_1-\frac{1}{2}p_2)\cos^2\frac{1}{2}p_1\Psi(p_2)}{D(\omega p_1)[\omega-\frac{3}{2}+\frac{1}{2}\cos(K-p_1)+\frac{1}{2}\cos(K-p_2)+\frac{1}{2}\cos(K-p_1-p_2)]}, \tag{91}$$

where

$$\omega=E/2J, \tag{92}$$

and we have expressed the denominator in a symmetrical fashion.

We shall simply indicate the final form of the coupled integral equations in three dimensions. They have a rather forbidding appearance. Let

$$D_{ij}=\frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}d\boldsymbol{\theta}(\cos p_{1i}-\cos\theta_i)\cos\theta_j/[\tfrac{9}{2}-\tfrac{1}{2}\sum_l\cos(K-p_1)_l-\sum_l\cos\tfrac{1}{2}p_{1l}\cos\theta_l-\omega] \tag{93}$$

$[i,j=(1,2,3)\equiv(x,y,z)]$. The summation over l implies that l is to be replaced by x, y, and z. D_{ij} is a (3×3) matrix. Define a kernel function

$$\begin{aligned}K_x(p_1p_2)=\{&(\cos\tfrac{1}{2}p_{1x}-\cos q_x)[(1+D_{22})(1+D_{33})-D_{23}D_{32}]+(\cos\tfrac{1}{2}p_{1y}-\cos q_y)\\&\times[D_{13}D_{32}-D_{12}(1+D_{33})]+(\cos\tfrac{1}{2}p_{1z}-\cos q_z)[D_{12}D_{23}-(1+D_{22})D_{13}]\}/\det(\delta_{ij}+D_{ij}),\end{aligned} \tag{94}$$

with

$$\mathbf{q}=\mathbf{K}-\mathbf{p}_1-\tfrac{1}{2}\mathbf{p}_2, \tag{95}$$

we have

$$T^1=\cos q_x\Psi_x(p_1)+\cos q_y\Psi_y(p_1)+\cos q_z\Psi_z(p_1), \tag{96}$$

and the equation for Ψ_x is

$$\Psi_x(p_1)=\frac{2}{(2\pi)^3}\int_{-\pi}^{\pi}d\mathbf{p}_2K_x(\mathbf{p}_1,\mathbf{p}_2)\sum_l\cos q_l\Psi_l(p_2)/\{\omega-\tfrac{1}{2}9+\tfrac{1}{2}\sum_l[\cos(K-p_2)_l+\cos\tfrac{1}{2}p_{2l}\cos q_l]\}. \tag{97}$$

Here, $\mathbf{q}$ is given by (95). The equations for Ψ_y and Ψ_z can be written down by inspection of (97) and (94). We shall not discuss these three-dimensional equations any further.

V. THREE-SPIN BOUND STATE IN ONE DIMENSION

Even Eq. (91) is sufficiently difficult for any analytical solution. However, a numerical method is suitable. We replace the integral by a sum to get a set of linear equation, and the bound-state solution is obtained as the zero of the determinant of this set of equations. For each fixed value of the c.m. momentum K, the determinant can be plotted against ω to find the zeros. Actually it is easy to guess the eigenvalue. When the three inverted spins move together as a unit, the dispersion law must be $E\sim(1-\cos K)$, as shown by (41) and (43). The exact numerical factor can be fixed by induction to be $\frac{1}{3}$.

$$E=\tfrac{1}{3}J(1-\cos K),\quad \omega=\tfrac{1}{6}(1-\cos K). \tag{98}$$

Bethe[2] verified that this was the correct eigenvalue. In the absence of any knowledge of the function $\Psi(p)$, we proceed as follows. Write (91) as

$$\Psi(p_1)=\lambda\int_{-\pi}^{\pi}\mathcal{K}(K,p_1,p_2)\Psi(p_2)dp_2, \tag{99}$$

with ω replaced by (98). Replace the integral by a sum where the kernel $\mathcal{K}(K,p_1,p_2)$ is obtained from (91)

[10] A. Ahmadzadeh and J. A. Tjon, Phys. Rev. **139**, B1085 (1965).

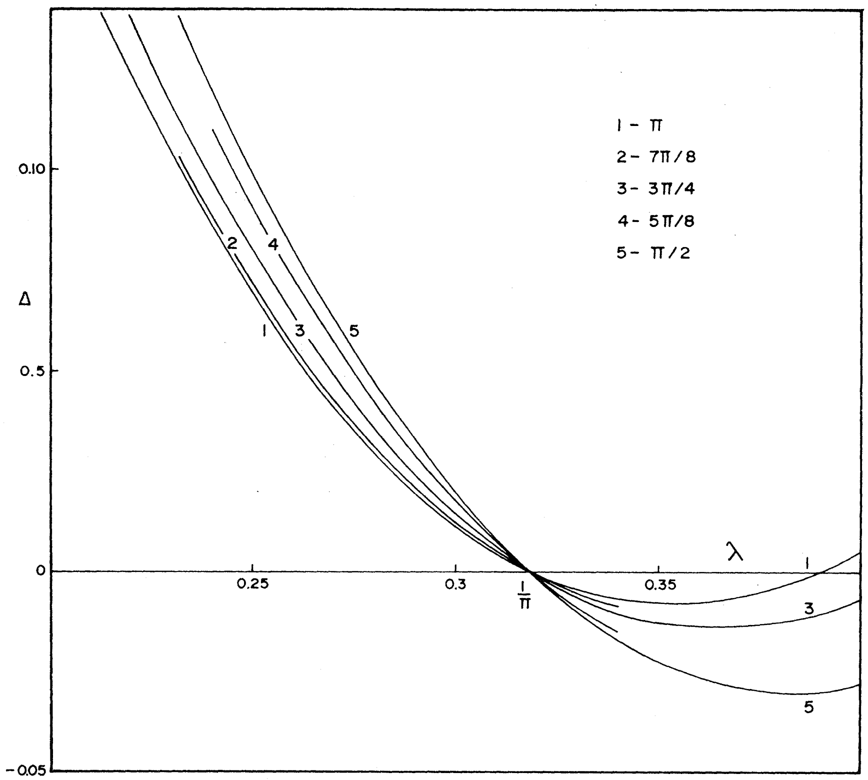

FIG. 1. Plot of the determinant Δ as a function of λ. The zero occurs for the correct eigenvalue $\lambda=1/\pi$. The inset refers to the values of K.

and evaluate the determinant of the set of linear equation, for each K, as function of λ. The determinant Δ must vanish for $\lambda=1/\pi$, no matter what K is, if the determinant size is sufficiently large so as to approximate the integral equation well enough. Figure 1 represents the result of a 32×32 determinant showing the expected behavior. For small values of K ($K<\frac{1}{2}\pi$), the proximity of the continuum decreases the accuracy, but there is a zero still found in the neighborhood of $1/\pi$. Accuracy here could have been improved by having a larger determinant.

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